NAME:

Math 351 Exam 1

Instructions: WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. SHOW YOUR WORK!

1. True, False, or incoherent
   a) If A is a subset of the real numbers, then sup(A) is a real number. [Hint: Be careful!] [2 pts]

   b) The field of complex numbers can be made into an ordered set. [2 pts]

   c) Every ordered set that has the least upper bound property also has the greatest lower bound property. [2 pts]

   d) Suppose that x = 0.101001000100001… is an infinite expansion in base 10. Then x has another representation in base 10. [2 pts]

   e) No number in (0, 1) has more than two decimal expansions in base p. [2 pts]

2. True, False, or incoherent
   a) A finite Cartesian product of countable sets is always countable. [2 pts]

   b) A countable Cartesian product of countable sets is countable. [2 pts]

   c) The cardinality of the power set $P(\mathbb{R})$ is bigger than card(A), where A is the set of all functions $f\colon \mathbb{R} \to \{0, 1\}$ [2 pts]

   d) Every infinite set has a proper subset of the same cardinality. [2 pts]

e) Every irrational number is a root of some polynomial with integer coefficients. [2 pts]

3. True, False, or incoherent

a) If d and p are metric functions on M, then so is $\sigma = \sqrt{d + p}$.
[2 pts]

b) If ℝ is equipped with the discrete metric, then diam (0, 4) = 4.
[2 pts]

c) {1/n} is a Cauchy sequence. [2 pts]

d) Every Cauchy sequence is convergent. [2 pts]

e) Equivalent metrics preserve Cauchy sequences. That is, if d and p are equivalent on M and $\{x_n\}$ is a sequence in M, then $\{x_n\}$ is Cauchy under the metric d if and only if $\{x_n\}$ is Cauchy under the metric p.
[2 pts]

4. True, False, or incoherent

a) An infinite intersection of open sets is never open. [2 pts]

b) All sets are either open or closed. [2 pts]

c) If ℝ is equipped with the discrete metric, the set (0, 1) is closed.
[2 pts]

d) Let F be a subset of ℝ, then the set of all limit points of F, $F'_{regular}$, under the regular metric is the same as F'_d — the set of all limit points of F under the metric $d(x, y) = \dfrac{|x-y|}{1+|x-y|}$. [2 pts]

e) The nested interval theorem guarantees that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ for any sequence of nested intervals $I_1 \supset I_2 \supset I_3 \supset ... \supset I_n \supset ....$ of $\mathbb{R}$.

[2 pts]

5. Let $0 < \alpha < 1$. Show that if x and y are positive real numbers, then $\left|x^{\alpha} - y^{\alpha}\right| \leq \left|x - y\right|^{\alpha}$. [Hint: $d(x, \ y) = \left|x - y\right|^{\alpha}$ defines a metric on $\mathbb{R}$]

[10 pts]

6. Let M = (0, ∞) be supplied with the metric function $d(x,\ y) = \left|\tan^{-1} x - \tan^{-1} y\right|$ and let $\{n\}_{n=1}^{\infty}$ be a sequence of positive integers.

   a) Is the sequence $\{n\}_{n=1}^{\infty}$ a Cauchy sequence in (M, d)? Justify your answer.

   [6 pts]

   b) Does the sequence $\{n\}_{n=1}^{\infty}$ converge in (M, d)? [4 pts]

7. Let (M, d) be a metric space. Prove that an open ball of (M, d) is always an open set of (M, d) [10 pts]

8. Decide whether the set $\bigcup_{n=1}^{\infty}[4n,\ 4n+1]$ is closed, open, or neither as a subset of $\mathbb{R}$. Justify your answer. [10 pts]

9. Let A = {a, b, c}. Define $F : A \to P(A)$ by

$$F(x) = \begin{cases} \{a,\ b,\ c\} & if \quad x = a \\ \{a,\ b\} & if \quad x = b \\ \{c\} & if \quad x = c \end{cases}$$

Compute $S_F = \{x \in A;\ x \notin F(x)\}$ [10 pts]

10. Decide whether ¾ is an element of the Cantor set Δ. Justify your answer. [10 pts]

11. Let $f : \Delta \to [0,1]$ be the Cantor function and let x, y $\in$ Δ with x < y. Show that $f(x) \le f(y)$. If $f(x) = f(y)$, show that x has two distinct binary decimal expansions. Finally, show that $f(x) = f(y)$ if and only if x and y are “consecutive” endpoints of the form $x = 0.a_1 a_2 ... a_n 1$ and $y = 0.a_1 a_2 ... a_n 2$ (base 3). [10 pts]